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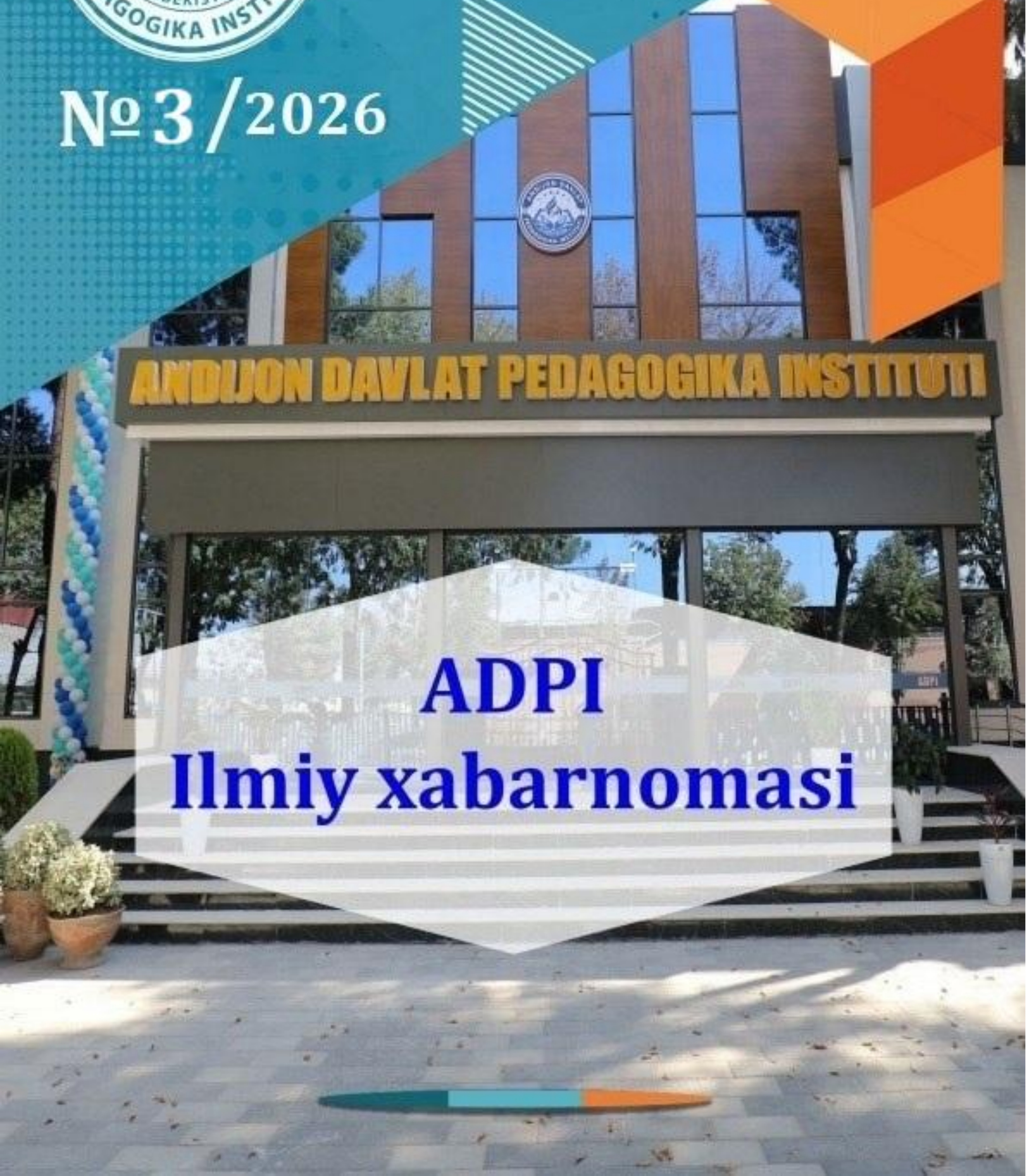

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ON A LINEAR PURSUIT DIFFERENTIAL GAME IN R^n

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Annotatsiya.
Ushbu maqolada quvish differensial oʻyini koʻrib chiqiladi. Oʻyinchilarning harakatlari birinchi tartibli chiziqli differensial tenglamalar bilan tavsiflanadi. Oʻyinchilarning boshqaruv funksiyalari integral chegaralanishlar qoʻyiladi. Maqolaning asosiy natijasi quvlovchi quvishni chekli vaqt oraligʻida yakunlashi uchun strategiya va kafolatlangan tutish vaqti uchun formula topishdir.

Kalit soʻzlar:
quvlovchi, qochuvchi, quvish differensial oʻyini, boshqaruv, kafolatlangan tutish vaqti, strategiya, quvishni yakunlash.

Аннотация.
В настоящей статье рассматривается дифференциальная игра преследования в пространстве R^n. Движения игроков описываются линейными дифференциальными уравнениями первого порядка. На функции управления игроков накладываются интегральные ограничения. Основным результатом данной статьи является нахождение стратегии преследователя для завершения преследования на конечном интервале времени и формула гарантированного времени преследования.

Ключевые слова:
Преследователь, убегающий, дифференциальная игра преследования, управление, гарантированное время преследования, стратегия, завершение преследования.

Abstract.
In the present paper, a pursuit differential game in the space R^n, is considered. The motions of the players are described by first order linear differential equations. The control functions of the players are subject to integral constraints. The main result of this paper is to find a strategy for the pursuer to complete the pursuit on a finite time interval and a formula for the guaranteed pursuit time.

Keywords:
Pursuer, evader, pursuit differential game, control, guaranteed pursuit time, strategy, completion of pursuit.

Introduction. In the mid-20th century, R.Isaacs pioneered the field of "Differential Games" to solve conflict problems. Subsequently, fundamental results were published in works such as [4, 5, 7-9]. More recently, differential game theory has focused primarily on examining scenarios where player control functions are subject to geometric, integral, or mixed constraints (e.g., [1-3, 6, 10-14]).

G. Ibragimov [2] studied a simple evasion differential game involving multiple pursuers and one evader. The maximum speed of pursuers is 1 and that of evader is . A new evasion strategy was proposed, demonstrating that the evader can avoid pursuers moving within any ε-vicinity of a straight line passing through the evader's initial state. In [11], a pursuit–evasion differential game involving one inertial pursuer and one inertial evader is studied. The pursuer's control is limited by integral constraints, while the evader's control is restricted by geometric constraints. In this study, a parallel approach strategy was proposed for the pursuit game, and sufficient conditions for its completion were obtained. Additionally, a sufficient condition for evasion was derived for the evasion game.

The work [3] investigated a differential game with multiple pursuers and a single evader, incorporating integral constraints and focusing on the optimal approach strategy. Player dynamics were described by linear systems of the same type. The game had a fixed duration, with its payoff functional defined as the minimum distance between the evader and the pursuers at termination. An estimate for this payoff functional was obtained.

The research methodology of this study is based on the theory of differential games and optimal control. The object of the research is a system of a pursuer and an evader moving in opposition within the space $\mathbb{R}^n$. Their dynamics are modeled using first-order linear differential equations, where the control functions are subject to integral constraints. At the initial stage, a precise mathematical model of the problem is formulated and the main assumptions are clearly defined. Then, analytical methods are applied to investigate the construction of a pursuit strategy. By employing auxiliary functions and estimation techniques, a guaranteed strategy for the pursuer is developed. In addition, the properties of the solutions of the differential equations are analyzed to derive sufficient conditions for successful pursuit.

The results are justified using rigorous mathematical proof techniques, and an explicit formula for the guaranteed pursuit time is obtained. Finally, the findings are generalized and compared with existing approaches in the field of differential games.

Results and Discussions. We consider the following differential game

$$\begin{aligned} \dot{x}(t) &= -ax(t) + u(t), \quad x(0) = x_0, \\ \dot{y}(t) &= -by(t) + v(t), \quad y(0) = y_0, \end{aligned} \tag{1}$$

where $t \geq 0, x(t), u(t), y(t), v(t), x_0, y_0 \in \mathbb{R}^n, x_0 \neq y_0, 2b > a > 0$ and $u(t)$ is the control parameter of the pursuer and $v(t)$ is that of the evader.

Definition 1. *Measurable functions* $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))$ *and* $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))$ *that satisfy the following integral constraints*

$$\int_0^\infty |u(s)|^2 ds \leq \rho^2, \tag{2}$$

$$\int_0^\infty |v(s)|^2 ds \leq \sigma^2 \tag{3}$$

are called admissible controls of the pursuer and evader respectively, where ρ *and* σ *are the given positive numbers.*

It is easy to check that the solution of equation (1) corresponding to admissible controls, is

$$x(t) = x_0 e^{-at} + \int_0^t e^{-a(t-s)} u(s) ds,$$

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$$y(t) = y_0 e^{-bt} + \int_0^t e^{-b(t-s)} v(s) ds.$$

Hence, at the time $\theta \geq 0$, the attainability set of the pursuer from the initial state x_0 is the ball $B\left(x_0 e^{-a\theta}, \varphi(a,\theta)\rho\right)$ of radius $\varphi(a,\theta)\rho$ and centred at $x_0 e^{a\theta}$, where

$$\varphi(a,\theta) = \left(\int_0^\theta e^{-2a(\theta-s)} ds\right)^{1/2} = \sqrt{\frac{1}{2a}(1 - e^{-2a\theta})}.$$

Indeed, by (2) and using Cauchy-Schwartz inequality we obtain

$$\begin{aligned} |x(\theta) - x_0 e^{-a\theta}| &\leq \int_0^t e^{-a(t-s)} |u(s)| ds \\ &\leq \left(\int_0^t e^{-2a(t-s)} ds\right)^{1/2} \cdot \left(\int_0^t |u(s)|^2 ds\right)^{1/2} \\ &\leq \varphi(a,\theta)\rho. \end{aligned}$$

In the same manner we can see that at the time $\theta \geq 0$, the attainability set of the evader from the initial state y_0, is the ball $B\left(y_0 e^{-b\theta}, \varphi(b,\theta)\sigma\right)$.

Definition 2. *Measurable function*

$$U(t,x,y,v), \quad U: [0,\infty) \times \mathbb{R}^{3n} \to \mathbb{R}^n,$$

is called strategy of the pursuer if, for any admissible control $v(t)$ of the evader, the initial value problem (1) has a unique solution $(x(t), y(t))$ at $u = U(t,x,y,v)$ and $v = v(t)$, and along this solution the following inequality holds

$$\int_0^\infty |U(s, x(s), y(s), v(s))|^2 ds \leq \rho^2.$$

Definition 3. *In the game (1), the pursuit is said to be completed within the time $\theta > 0$, if there exists admissible strategy $U(t,x,y,v)$ of pursuer such that $x(t) = y(t)$ holds for any admissible control $v = v(t)$ of the evader, for some $t \in [0,\theta]$*

The main result of this section is stated by following theorem.

Theorem. *If $(2b - a)\rho^2 > b\sigma^2$, then pursuit can be completed for the finite time $\theta > 0$ in the game (1).*

Proof. To prove the theorem we first find a formula for a guaranteed pursuit time. Then we show the pursuit can be completed in the game by choosing a control to pursuer.

A formula for a guaranteed pursuit time. Consider the following equation for $t > 0$,

$$\frac{\varphi^2(b,t)}{\varphi(2b-a,t)}\rho = |x(0)e^{-at} - y(0)e^{-bt}| + \varphi(b,t)\sigma, \tag{4}$$

where

$$\varphi(2b-a,t) = \left(\int_0^t e^{-2(2b-a)(t-s)}ds\right)^{1/2} = \sqrt{\frac{1}{2(2b-a)}(1-e^{-2(2b-a)t})}.$$

The equation (4) has a root in the interval $(0,+\infty)$ since if $t \to 0^+$, then (4) turns to the inequality $0 < |x(0) - y(0)| + 0$, (here $t \to 0^+$ denotes the right-hand limit), and if $t \to +\infty$, then (4) turns to the inequality $\frac{\rho}{\sqrt{b}} > 0 + \frac{\sigma}{\sqrt{2b-a}}$, which is correct by the assumption of the theorem.

Let $t = \theta > 0$ be the minimum root of the equation (4) in the interval $(0,+\infty)$.

In the inequality

$$\int_0^\theta e^{-b(\theta-s)}|v(s)|ds \le \left(\int_0^\theta e^{-2b(\theta-s)}ds\right)^{\frac{1}{2}} \cdot \left(\int_0^\theta |v(s)|^2 ds\right)^{\frac{1}{2}},$$

the equality sign occurs when

$$|v(s)| = ke^{-b(\theta-s)}, \quad 0 \le s \le \theta, \quad k = const. \tag{5}$$

If the evader spends energy equal to $\int_0^\theta |v(s)|^2 ds = \sigma^2$, then using equation (5) we find $k = \frac{\sigma}{\varphi(b,0,\theta)}$. Hence, we have

$$|v(s)| = \frac{\sigma}{\varphi(b,0,\theta)}e^{-b(\theta-s)}, \quad 0 \le s \le \theta. \tag{6}$$

Further, we will use the control

$$u_0(s) = \frac{y_0e^{-b\theta} - x_0e^{-a\theta}}{\varphi^2(b,\theta)}e^{(a-2b)(\theta-s)}, \tag{7}$$

which transfers the state of the pursuer from the point $x_0e^{-a\theta}$ to $y_0e^{-b\theta}$ at time θ. Indeed,

$$\begin{aligned} x(\theta) &= x_0 e^{-a\theta} + \int_0^\theta e^{-a(\theta-s)} u_0(s) ds \\ &= x_0 e^{-a\theta} + \frac{y_0 e^{-b\theta} - x_0 e^{-a\theta}}{\varphi^2(b,\theta)} \int_0^\theta e^{-2b(\theta-s)} ds \\ &= x_0 e^{-a\theta} + y_0 e^{-b\theta} - x_0 e^{-a\theta} = y_0 e^{-b\theta}. \end{aligned}$$

The strategy for the pursuer. Let the pursuer applies the strategy defined by

$$\overline{u}(s) = \begin{cases} u_0(s) + e^{(b-a)(\theta-s)} v(s), & 0 \le s \le \theta, \\ 0, & s > \theta. \end{cases} \tag{8}$$

We are now position to show that the strategy (8) is admissible. Application of Cauchy-Schwartz inequality gives that

$$\left(\int_0^\infty |\overline{u}(s)|^2 ds\right)^{1/2} = \left(\int_0^\theta |\overline{u}(s)|^2 ds\right)^{1/2}$$
$$\le \left(\int_0^\theta |u_0(s)|^2 ds\right)^{1/2} + \left(\int_0^\theta e^{2(a-b)(\theta-s)} |v(s)|^2 ds\right)^{1/2}.$$

By applying (6) and (4) in turn, we get

$$\begin{aligned} &\left(\int_0^\theta \frac{|y_0 e^{-b\theta} - x_0 e^{-a\theta}|^2}{\varphi^4(b,\theta)} e^{-2(2b-a)(\theta-s)} ds\right)^{1/2} \\ &+ \left(\int_0^\theta \frac{\sigma^2}{\varphi^2(b,\theta)} e^{-2(2b-a)(\theta-s)} ds\right)^{1/2} \\ &= \frac{|y_0 e^{-b\theta} - x_0 e^{-a\theta}|}{\varphi^2(b,\theta)} \left(\int_0^\theta e^{-2(2b-a)(\theta-s)} ds\right)^{1/2} \\ &+ \frac{\sigma}{\varphi(b,\theta)} \left(\int_0^\theta e^{-2(2b-a)(\theta-s)} ds\right)^{1/2} \\ &= \frac{|y_0 e^{-b\theta} - x_0 e^{-a\theta}|}{\varphi^2(b,\theta)} \varphi(2b-a,\theta) + \frac{\sigma}{\varphi(b,\theta)} \varphi(2b-a,\theta) = \rho. \end{aligned}$$

This implies that the strategy (8) is admissible.

Completion of pursuit. We now prove that the pursuit can be completed at the time $t = \theta$. We let the pursuer applies the strategy (8). Consequently we obtain

$$\begin{aligned} x(\theta) &= x_0 e^{-a\theta} + \int_0^\theta e^{-a(\theta-s)} \left(\frac{y_0 e^{-b\theta} - x_0 e^{-a\theta}}{\varphi^2(a,\theta)} e^{-a(\theta-s)} + e^{(b-a)(\theta-s)} v(s) \right) ds \\ &= x_0 e^{-a\theta} + \int_0^\theta e^{-a(\theta-s)} \frac{y_0 e^{-b\theta} - x_0 e^{-a\theta}}{\varphi^2(a,\theta)} e^{-a(\theta-s)} ds \\ &\quad + \int_0^\theta e^{-a(\theta-s)} e^{(a-b)(\theta-s)} v(s) ds \\ &= x_0 e^{-a\theta} + \frac{y_0 e^{-b\theta} - x_0 e^{-a\theta}}{\varphi^2(a,\theta)} \int_0^\theta e^{-2a(\theta-s)} ds + \int_0^\theta e^{-b(\theta-s)} v(s) ds \\ &= x_0 e^{-a\theta} + y_0 e^{-b\theta} - x_0 e^{-a\theta} + \int_0^\theta e^{-b(\theta-s)} v(s) ds = y(\theta). \end{aligned}$$

This finishes the proof.

Example 5. Let the following differential game is given in $\mathbb{R}^2$,

$$\begin{aligned} \dot{x}(t) &= -2x(t) + u(t), \quad x_0 = (0,0), \\ \dot{y}(t) &= -\tfrac{3}{2} y(t) + v(t), \quad y_0 = (0,10), \end{aligned}$$

where $t \geq 0, x(t), u(t), y(t), v(t) \in \mathbb{R}^2$ and the control parameters of the players satisfy following constraints, respectively

$$\int_0^\infty |u(s)|^2 ds \leq 1.21, \quad \int_0^\infty |v(s)|^2 ds \leq 1,$$

that is, $\rho = 1.1$, $\sigma = 1$, $a = 2$ and $b = 3/2$. By the formula (4), the guaranteed pursuit time $\theta \approx 2.19$. In this game, the pursuer completes the pursuit at the time θ using the following strategy

$$\overline{u}(s) = \begin{cases} u_0(s) + e^{-\frac{1}{2}(\theta-s)} v(s), & 0 \leq s \leq \theta, \\ 0, \quad s > \theta, \end{cases}$$

where

$$u_0(s) = \frac{3e^{\left(\frac{1}{2}\theta + s\right)}}{e^{3\theta} - 1} y_0.$$

Conclusion. This paper has investigated a pursuit differential game between two players, each governed by a distinct first-order linear differential equation. Our primary focus was identifying conditions for the pursuer to successfully complete the pursuit process. By imposing appropriate constraints on both players' control resources, we established sufficient conditions guaranteeing pursuit completion.

An explicit formula for the guaranteed pursuit time was also derived, quantitatively characterizing the process. Based on this result, we proposed a constructive strategy for the pursuer and rigorously proved that it ensures pursuit completion within a finite time horizon. The analysis highlights the crucial role of control resource allocation and system dynamics in determining the game's outcome.

Finally, a specific example illustrated these theoretical findings, demonstrating the developed approach's applicability and confirming the results' validity in a particular case. The study's results contribute to differential game theory and may inform further research in control theory and related applied fields.

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